International Journal of Theoretical Physics, Vol. 30, No. 11, 1991

Hamilton's Principle, Covariant Hamiltonian, and Canonical Formalism in Four and Five Dimensions

Ludger Hannibal[1]

Received November 22, 1990

A discussion of the 1950s and 1960s on the existence of an explicit covariant canonical formalism is renewed. A new point of view is introduced where Hamilton's principle, based on the existence of a Hamiltonian, is postulated independently from the Lagrange formalism. The Hamiltonian is determined by transformation properties and dimensional considerations. The variation of the action without constraints leads to an explicit covariant canonical formalism and correct equations of motion. The introduction of the charge as a fifth momentum gives rise to a reformulation of classical relativistic point mechanics as a five-dimensional $U(1)$ gauge theory with a theoretically invisible extra dimension. A generalization to other gauge groups is given. The inversion of the proper time is introduced as a new particle–antiparticle symmetry that allows one to show that in the five-dimensional classical theory all particles have positive energy.

1. INTRODUCTION

The existence of a Lagrange function and Hamilton's variational principle that leads to the Euler–Lagrange equations of motion form the generally accepted basis of classical point mechanics. In nonrelativistic classical mechanics the definition of the Hamiltonian as a transformation of the Lagrangian leads to the equivalent canonical equations that may be derived from the variational principle with a Hamiltonian action. This situation changes in classical relativistic mechanics. The generally accepted manifest Lorentz-invariant Lagrangian for relativistic point particles is given by (Arzeliès, 1972; Doughty, 1990)

$$L(x^\mu, \dot{x}^\mu, \tau) = m_0 c[\dot{x}^\mu g_{\mu\nu}(x)\dot{x}^\nu]^{1/2} + qA_\mu(x)\dot{x}^\mu \tag{1}$$

[1]Institut für Theoretische Physik, Universität Düsseldorf, D-4000 Düsseldorf, Germany.

0020-7748/91/1100-1431$06.50/0

but there is no equivalent covariant canonical formalism, since, if we define the vector of the four-momentum by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m_0 c \frac{g_{\mu\nu}\dot{x}^\nu}{(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{1/2}} + qA_\mu \tag{2}$$

the transformation from the velocities to the momenta is singular due to the constraint

$$\left(\frac{\partial L}{\partial \dot{x}^\mu} - qA_\mu\right) g^{\mu\nu} \left(\frac{\partial L}{\partial \dot{x}^\nu} - qA_\nu\right) = m_0{}^2 c^2 \tag{3}$$

and we only arrive at the implicit equation

$$H\left(x^\mu, \frac{\partial L}{\partial \dot{x}^\mu}\right) = 0 \tag{4}$$

but not at an explicit canonical formalism. The implicit formalism with arbitrary constraints in phase space was developed by Dirac (1950, 1958). There followed a discussion on the existence of an explicit covariant canonical formalism based on a Hamiltonian in the 1950s and 1960s (Arzeliès, 1972; Macke, 1952; Falk, 1952; Sauter, 1959; Stephani, 1962; Schay, 1962; Rund, 1962; Kalman, 1962; Linder, 1965), with the above result (Linder, 1965) that if we start with a covariant Lagrangian, then no explicit Hamiltonian exists due to the fact that the Lagrangian is homogeneous of degree one. Previously, Falk (1952) had given an explicit Hamiltonian. He argued that the constraint

$$\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu = c^2 \tag{5}$$

together with the desired equation

$$\dot{x}^\mu = \frac{\partial M}{\partial p_\mu} \tag{6}$$

allows for M the complete integral

$$M = [(p_\mu - \chi_\mu) g^{\mu\nu} (p_\nu - \chi_\nu)]^{1/2} \tag{7}$$

which is basically the rest mass of the particle, and he showed that if χ_μ is identified with the electromagnetic potential, then the canonical equations, assumed to hold, lead to the correct equations of motion. He thus ignored the dependence by definition of the Hamiltonian on the Lagrangian. For this reason his theory was rejected and even denied physical significance by Schay (1962). But it is obvious that Falk's theory does give correct physical results. We now intend to put these results on a well-defined basis. We do this by postulating the existence of a Hamiltonian and the corresponding action principle independently from the Lagrangian. We then determine the Hamiltonian of this theory from transformation properties and dimensional considerations and thus avoid the heavily criticized use of some of the desired properties of the theory, which we instead will then be able to *derive*, and, going beyond Falk's theory, will elucidate other intriguing properties that the Lagrangian formalism does not feature, including the construction of a five-dimensional generalization.

2. THE POSTULATE ON THE HAMILTONIAN

We now formulate a postulate, called Postulate H, that we think incorporates the minimal assumptions one has to make in order to set up a theory of a single classical relativistic point particle in external fields. Where necessary we give alternate formulations for special relativity, marked by (S), and general relativity, marked by (G). In special relativity the metric $\boldsymbol{\eta}$ of space-time is assumed to be Minkowskian and the gravitational field $\mathbf{g}$ is different from $\boldsymbol{\eta}$, whereas in general relativity $\mathbf{g}$ is the metric of the space-time manifold, $\mathbf{g}=\boldsymbol{\eta}$ only in the absence of gravitation. We are thus open for different theories of gravitation, noting that the theory for a single point particle is independent of the geometry of the space-time manifold as long as we do not consider the action for the fields. The basic difference arises in the set of admissible coordinate transformations.

2.1. Postulate H

A point particle is described by its position $(\mathbf{x}(\tau), \mathbf{p}(\tau))$ in the cotangent bundle of a four-dimensional space-time manifold M, where (S) M is $\mathbb{R}^4$, (G) a four-dimensional pseudo-Riemannian space-time manifold.

The space-time coordinates x^μ, $\mu=0, 1, 2, 3$, have physical dimension of length, the momentum coordinates p_μ, $\mu=0, 1, 2, 3$, of the dual of the tangent space that of mass times velocity, and the parameter τ that of time. The time is given by $t=x^0/c$, c being the constant velocity of light. Under coordinate transformations $x^\mu \to x^{\mu'}(x^\nu)$ the coordinates p_ν transform as a covariant four-vector, the electromagnetic potential $A_\mu(\mathbf{x})$ is a covariant tensor of first rank, and the (S) gravitational field, (G) the metric $g_{\mu\nu}(\mathbf{x})$ is a covariant tensor of second rank.

A Hamiltonian $H(\mathbf{x}, \mathbf{p}, \tau)$ exists that is defined on the eight-dimensional phase space and is a function of a real parameter τ, with the following properties:

H is real-valued and has the physical dimension of energy. H as a function of the coordinates transforms as a scalar under the admissible set of transformations, which are (S) the Poincaré group, (G) general coordinate transformations. For physical trajectories $(\mathbf{x}(\tau), \mathbf{p}(\tau))$, $\tau_0 \leq \tau \leq \tau_1$, of a particle the variation of the action

$$S_H = \int_{\tau_0}^{\tau_1} \{\mathbf{p}(\tau)\, d\mathbf{x}(\tau) - H(\mathbf{x}(\tau), \mathbf{p}(\tau), \tau)\, d\tau\}$$

vanishes, $\delta S_H = 0$, for any independent variations $\delta\mathbf{x}$ and $\delta\mathbf{p}$ that vanish at the limits τ_0 and τ_1.

We note that the use of the cotangent bundle gives a natural interpretation to the "product" $\mathbf{p}\, d\mathbf{x}$. The coordinates are assumed to be canonical in that $\mathbf{p}\, d\mathbf{x} = p_\mu\, dx^\mu$ in any coordinates. We now analyze general implications of this postulate. From the variational principle the equations of motion are derived. The variation of the action, written in coordinates, is given by

$$\delta S_H = \int_{\tau_0}^{\tau_1} \left[\left\{ \frac{\partial}{\partial x^\kappa} (p_\nu \dot{x}^\nu - H) \right\} \delta x^\kappa + \left\{ \frac{\partial}{\partial p_\kappa} (p_\nu \dot{x}^\nu - H) \right\} \delta p_\kappa \right] d\tau \qquad (8)$$

since the variations δx^κ and δp_κ are independent. The condition $\delta S_H = 0$ yields, with a partial integration, using that the variations vanish at the limits,

$$\dot{p}_\kappa = -\frac{\partial H}{\partial x^\kappa}$$
$$\dot{x}^\kappa = \frac{\partial H}{\partial p_\kappa} \qquad (9)$$

So Postulate H leads to Hamilton's canonical equations and hence to a full canonical formalism with Hamilton–Jacobi differential equations and Poisson brackets. Since the derivation of the equations of motion is independent of the chosen coordinates, it immediately follows that coordinate transformations are canonical transformations. If some coordinate x^λ is cyclic, then the corresponding momentum p_λ is a constant of motion. Independence from time thus leads to energy conservation, the energy being given by cp_0. If H does not explicitly depend on the parameter τ, $\partial H/\partial\tau = 0$, we have

$$\frac{d}{d\tau} H = \frac{\partial H}{\partial x^\kappa} \dot{x}^\kappa + \frac{\partial H}{\partial p_\kappa} \dot{p}_\kappa = 0 \qquad (10)$$

Hence H itself is a constant of motion. Considering the dimension, we see that $c^{-2}H$ is a conserved mass that only can be the rest mass of the particle under consideration, up to a constant multiple. We thus will be able to derive the constancy of the rest mass from our theory if we can find a Hamiltonian independent of τ that satisfies Postulate H. This is a first major difference from Lagrangian theory. There one has to introduce the rest mass as a constant in order to build a Lagrangian with the correct physical dimension and there is no means to derive the constancy of the rest mass. We did not provide in Postulate H for such a constant because we can do without it, as we now proceed to show.

3. THE HAMILTONIAN

If we require that the Hamiltonian H_0 for a free particle neither depends on the parameter τ nor on the space-time coordinates, then from dimensional considerations we are left with the scalar

$$H_0 = c(p_\mu \eta^{\mu\nu} p_\nu)^{1/2} \tag{11}$$

and constant multiples of this. Any other Hamiltonian is excluded because we then have to introduce a constant with dimension of mass. This constant could be only the mass of a particle, leading to the familiar constraint $p^\mu p_\mu =$ const that is not incorporated in the theory. Such a theory is therefore rejected (Doughty, 1990). Our theory, on the contrary, will be a theory for particles of any mass, without any constraints. The free real number in (11) has been chosen such that $c^{-2}H_0$ indeed becomes the rest mass of the particle, which is just Falk's (1952) result. The magnitude of the rest mass is to be determined from the initial conditions $p_\mu(\tau_0)$, the Hamiltonian is the same for all particles. The canonical equations for the free particle read

$$\begin{aligned} \dot{p}_\kappa &= -\frac{\partial H}{\partial x^\kappa} = 0 \\ \dot{x}^\kappa &= \frac{\partial H}{\partial p_\kappa} = c((p_\mu \eta^{\mu\nu} p_\nu)^{1/2})^{-1} \eta^{\kappa\lambda} p_\lambda \end{aligned} \tag{12}$$

The first equation expresses the conservation of the four-momentum; from the second the relation

$$\dot{x}^\mu \eta_{\mu\nu} \dot{x}^\nu = c^2 \tag{13}$$

follows, which is just Falk's starting relation (5). Hence it is a *consequence* of our theory that the parameter τ is always the proper time of the particle. This is the second major difference from Lagrangian theory, where one has reparametrization invariance. It thus becomes clear that both formalisms *cannot* be equivalent. The Lagrangian is one function for particles of fixed constant rest mass and arbitrary parameterization of the trajectories, the Hamiltonian is one function for particles of any rest mass that is a constant of motion, whereas the parameter is fixed by the equations of motion to be the proper time. The Hamiltonian for a particle in an external gravitational and electromagnetic field is now obtained from the free Hamiltonian H_0 by the "minimal" substitution

$$\boldsymbol{\eta} \rightarrow \mathbf{g}, \qquad p_\mu \rightarrow p_\mu - qA_\mu(x) \tag{14}$$

where q is the electric charge of the particle. To be in accordance with Postulate H we can simply add "A particle possesses a constant electric charge q" to it. We did not provide for this in Postulate H because it is obvious that this introduces a feature we had avoided with respect to the rest mass; there are now different Hamiltonians for differently charged particles. We now first discuss the resulting theory and then will show in Section 4 that there exists a second possibility that leads to a five-dimensional theory without this undesirable feature.

The Hamiltonian for a particle with charge q and arbitrary mass is postulated to be given by

$$H(x^\mu, p_\nu) = c[(p_\mu - qA_\mu(x))g^{\mu\nu}(x)(p_\nu - qA_\nu(x))]^{1/2} \tag{15}$$

Again the rest mass of the particle, given by $m_0 = c^{-2}H$ is a constant of motion. We use this to write the equations of motion as

$$\begin{aligned} \dot{x}^\mu &= m_0{}^{-1} g^{\mu\nu}(p_\nu - qA_\nu) \\ \dot{p}_\mu &= q\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu - \frac{1}{2}m_0\frac{\partial g_{\kappa\lambda}}{\partial x^\mu}\dot{x}^\kappa\dot{x}^\lambda \end{aligned} \tag{16}$$

Again the parameter τ is fixed to be the proper time, now defined by

$$\dot{x}^\mu g_{\mu\nu}(x)\dot{x}^\nu = c^2 \tag{17}$$

We may eliminate p_μ from the equations of motion to obtain the more familiar form

$$m_0\ddot{x}^\mu = m_0 G^\mu + qF^\mu \tag{18}$$

where

$$G^\mu = \frac{1}{2}g^{\mu\nu}\left[\frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\kappa}}{\partial x^\lambda} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa}\right]\dot{x}^\kappa\dot{x}^\lambda \tag{19}$$

is the force of the gravitational field and

$$F^\mu = g^{\mu\nu}\left[\frac{\partial A_\nu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\nu}\right]\dot{x}^\lambda \tag{20}$$

is the electromagnetic force.

So we have recovered the well-known equation (18) from Postulate H with the Hamiltonian given by (15). We moreover point out that it follows from equations (17) and (18) together that the inertial mass and the gravitational mass of a point particle are equal. This becomes explicit if we rewrite equation (18) in terms of the observable velocity

$$v^\mu(t) = \frac{d}{dt} x^\mu(\tau(t)) = \dot{x}^\mu(\tau(t)) \frac{d\tau}{dt}(t) \tag{21}$$

and the relativistic mass

$$m = m_0 |dt/d\tau| = m_0 c/(v^\nu g_{\mu\nu} v^\nu)^{1/2} \tag{22}$$

We obtain

$$\frac{d}{dt} m v^\mu = -m\Gamma^\mu{}_{\nu\lambda} v^\nu v^\lambda + \sigma q F^\mu{}_\nu v^\nu \tag{23}$$

The sign

$$\sigma = \frac{dt/d\tau}{|dt/d\tau|} \tag{24}$$

is related to particles and antiparticles, as we will see in Section 5. We clearly see that *inertial mass equals gravitational mass equals relativistic mass.* We may well say that this is a law within the theory because in writing down the Hamiltonian we did not make any assumption about any of these masses. The relativistic mass is uniquely fixed only by the presence of the electromagnetic force, that is, by the value of q. We see that a particle moves as if **g** were the metric and equation (27) tell us that $c\, d\tau$ is the line element of the presumed metric **g**, but in fact we do not need to assume this as long as we consider only the motion of a single particle in external fields. Phenomena such as the advance of planetary perihelia, the bending of light, the gravitational red shift, and the time delay of signals are derived from the equations of motion and are a test for the values of the gravitational field (Anderson, 1967). These equations depend on the coupling of the gravitational field to the action; with respect to the generation of this field we are still open to any theory consistent with the equations of motion (Anderson, 1967).

4. FIVE DIMENSIONS AND MORE

4.1. Five-Dimensional Reformulation of Classical Relativistic Mechanics as a $U(1)$-Gauge Theory

The construction of the Hamiltonian for charged particles on the eight-dimensional phase space has brought the strange feature that we have the same Hamiltonian for particles of any mass, but different functions if the charge is different. It can be seen that it is the introduction of a fourth pair of coordinates into the relativistic theory compared to the nonrelativistic case that allowed us to eliminate from the Hamiltonian the constant rest mass, which instead became a constant of motion. This suggests that instead of postulating a constant electric charge, we could introduce a fifth pair (x^4, p_4) of coordinates and try to obtain the charge as another constant of motion. Obviously the momentum p_4 is conserved if the coordinate x^4 is cyclic. Moreover, it is immediately seen that if we directly replace the charge q in the Hamiltonian (15) by the momentum p_4

$$q \to p_4 \tag{25}$$

then the Hamiltonian becomes the root of a quadratic form in the five-momentum,

$$H \to \hat{H}(x^{\hat{\mu}}, p_{\hat{\mu}}) = c[p_{\hat{\mu}} \hat{g}^{\hat{\mu}\hat{\nu}}(x^\kappa) p_{\hat{\nu}}]^{1/2}, \qquad \hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4 \tag{26}$$

with the five-dimensional contravariant second-rank tensor $\hat{\mathbf{g}}$ given by

$$\hat{\mathbf{g}} = \begin{bmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & A^\kappa A_\kappa \end{bmatrix} \tag{27}$$

This matrix is singular and hence not the inverse of some five-dimensional metric. This yields a theory equivalent to the four-dimensional theory with respect to particle motion, but it is not of Kaluza (1921)–Klein (1926) type. The fifth dimension is nongeometrical, it constitutes a true "internal" space. Due to this circumstance we are not forced to introduce a constant that makes the electromagnetic potential dimensionless; instead, we directly ascribe the dimension of electric charge to the fifth momentum that then may be measured in units of the elementary charge e, and the fifth coordinate x^4 attains the dimension of action divided by charge and may be measured in units of h/e, h the Planck constant. The theory is not geometric and avoids the use of the gravitational constant in the particle action.

We consequently assume that the ten-dimensional phase space is now given by the product of the original eight-dimensional phase space and the cotangent bundle of a one-dimensional (compact) manifold. The topology of the internal space is linked with charge quantization in a quantized theory (Klein, 1926; Souriau, 1963). The five-dimensional action reads

$$S_{\hat{H}} = \int_{\tau_0}^{\tau_1} \{\mathbf{p}(\tau)\, d\mathbf{x}(\tau) - \hat{H}(\mathbf{x}(\tau), \mathbf{p}(\tau))\, d\tau\} \tag{28}$$

This is one action for particles of any mass and any charge. The four-dimensional equations of motion (16) where we replace q by p_4, together with the new equations

$$\dot{p}_4 = 0$$
$$\dot{x}_4 = m_0^{-1} \hat{g}^{4\hat{\mu}} p_{\hat{\mu}} \tag{29}$$

constitute the set of the five-dimensional canonical equations. The particle motion is not changed by these equations; there is no new force and the parameter τ is still the proper time, a major difference from Kaluza–Klein theory. Due to the fact that the coordinate x^4 is cyclic, the charge $q = p_4$ is now, like the rest mass, a constant of motion, to be determined from the initial condition $p_4(\tau_0)$. Since the line element

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{30}$$

remains unchanged, length measurements in space-time are not influenced; there is no physical length associated with the new dimension, it is truly "internal" and theoretically invisible. This justifies the choice of the electric charge, which is not related to any length scale, as the physical dimension of the internal space. The second equation of motion in (29) seems insignificant at first sight, but it has an interesting consequence. The action vanishes, $S_{\hat{H}} = 0$, for any solution of the five-dimensional equations of motion inserted. This is a universal feature, independent of particle properties, trajectories, and external fields, whereas in any three- or four-dimensional theory the values of the action depend on these.

Our five-dimensional theory substantially differs from the original ideas of Kaluza (1921) and Klein (1926), since it is impossible to extend the Einstein–Hilbert action for the gravitational field to a unified action for the singular field $\hat{\mathbf{g}}$. A feature in common with Kaluza–Klein theories is that gauge transformations are induced by coordinate transformations of the type

$$x^{\mu\prime} = x^\mu, \qquad x^{4\prime} = x^4 + f(x^\mu) \tag{31}$$

The covariant transformations of $p_{\hat{\mu}}$ and $\hat{g}^{\hat{\mu}\hat{\nu}}$ are given by

$$g^{\mu\nu\prime} = g^{\mu\nu}, \qquad A^{\mu\prime} = A^\mu + g^{\mu\nu}\, \partial f/\partial x^\nu$$
$$p_\mu{}' = p_\mu - p_4\, \partial f/\partial x^\mu \qquad p_4{}' = p_4 \tag{32}$$

Hence the Hamiltonian and the action $S_{\hat{H}}$ are invariant under these gauge transformations, whereas the Lagrangian (1) and the action S_H are not, they change by a boundary term. If we assume that the internal space is topologically S^1 (Einstein and Bergmann, 1938), then we may speak of our theory as a $U(1)$ gauge theory since the coordinate transformations (31) are induced by the operation of the group $U(1)$ on itself (Souriau, 1963). If

$$\gamma(x^\mu) = \exp[if(x^\mu)e/h] \in U(1) \tag{33}$$

defines a local group element, the transformation (31) may be defined by

$$\exp(ix^{4\prime}e/h) = \gamma(x^\mu)\exp(ix^4 e/h) \tag{34}$$

where we have explicitly taken account of the physical dimensions.

4.2. Generalization to Other Gauge Groups

The generalization to other gauge groups is straightforward. We take the cotangent bundle of any N-dimensional compact Lie group G with canonical coordinates $y^j = x^{3+j}$ and p_{3+j}, $j = 1, \ldots, N$, as the internal phase space. As fields we introduce N four-vectors $B^{\mu j}(\mathbf{x})$ and the $(4+N)\times(4+N)$ matrix $\tilde{g}^{\tilde{\mu}\tilde{\nu}}$, $\tilde{\mu}$, $\tilde{\nu} = 0, 1, \ldots, 3+N$,

$$\tilde{g} = \begin{bmatrix} g^{\mu\nu} & -B^{\mu j} \\ -B^{\nu k} & B^{\lambda k} B_\lambda{}^j \end{bmatrix} \tag{35}$$

The Hamiltonian is defined by

$$\tilde{H} = c(p_{\tilde{\mu}}\tilde{g}^{\tilde{\mu}\tilde{\nu}}p_{\tilde{\nu}})^{1/2} \tag{36}$$

and the action as before. The matrix $\tilde{\mathbf{g}}$ has rank 4 and therefore again the proper time is defined by relation (17) and length measurements are not influenced by the additional dimensions. Thus we have the same topology, but not the same geometry as standard unified theories (Weinberg, 1983). The action and the Hamiltonian are now invariant under local gauge transformations that depend only on the four coordinates x^μ. Gauge transformations can be defined via the operation of the group $G \times G$ on G, as there are transformations corresponding either to the operation of the group on its left or on its right: Let L_j be a basis of the Lie algebra of G and $(\alpha(x^\mu), \beta(x^\mu)) \in G \times G$ be a local pair of group elements of G. Then a transformation of the coordinates

$$y^j \to y^{j\prime}(y^j, x^\mu)$$

is defined by

$$\exp[L_j y^{j\prime}(y^j, x^\mu)] = \alpha(x^\mu) \circ \exp(L_j y^j) \circ \beta(x^\mu) \tag{37}$$

This is well defined due to the properties of a Lie group and the canonical coordinates. Two transformations defined by $(\alpha_1(x^\mu), \beta_1(x^\mu))$ and

$(\alpha_2(x^\mu), \beta_2(x^\mu))$ are identical if

$$\alpha_1^{-1} \circ \alpha_2 = \beta_1 \circ \beta_2^{-1} = \gamma(x^\mu) \in C(G) \tag{38}$$

are elements of the center $C(G)$ of the gauge group G. This induces an equivalence relation $\sim$ on $G \times G$; the true local gauge group consists of the quotient $G \times G_{/\sim}$. If $\beta = \alpha^{-1}$, then the coordinates transform linearly under the adjoint representation of G. We may split any transformation (α, β) into a "rotation" (α, α^{-1}) and a "translation" $(1, \alpha \circ \beta)$. If the group G is Abelian, the gauge group is G itself and there are no rotations and the "translations" are true translations as in the $U(1)$ case. This splitting gives rise to a second mapping of $G \times G$ into the transformations on G, defined by

$$((\alpha, \beta), g) \mapsto \alpha \circ g \circ \alpha^{-1} \circ \beta \tag{39}$$

whence the quotient is taken only with respect to the first factor. This structure of the gauge group is interesting if one considers the groups $SU(3)_{\text{color}}$ and $SU(3)_{\text{flavor}}$ and the fact that the charges of quarks, multiples of one-third of the elementary charge, seem to be related to the center of the group $SU(3)$.

The fields transform like

$$\begin{aligned} g^{\mu\nu\prime} &= g^{\mu\nu} \\ B^{\mu j\prime} &= (\partial y^{j\prime}/\partial y^k) B^{\mu k} + g^{\mu\nu}(\partial y^{j\prime}/\partial x^\nu) \end{aligned} \tag{40}$$

The transformed fields in general will depend on the coordinates of the internal space. Hence not all momenta (charges) of this space are conserved if such transformations are physically allowed. If the Lie algebra contains the identity, then the corresponding charge is always conserved, as in the $U(1)$ case. On the other hand, if one allowed transformations of the four coordinates x^μ to depend on the internal coordinates, such a conservation law would be destroyed. Since such transformations leave the action invariant, criteria for the admissibility of coordinate transformations have to be found independently from the particle action and are beyond the scope of this work, since one has to consider the action for the fields.

5. PARTICLES AND ANTIPARTICLES

We now discuss the concept of antiparticles within the framework of the five-dimensional theory. The equation of motion (23), expressed in observable quantities, shows that it describes two kinds of particles. There are those with mass m and charge q if $dt/d\tau$ is positive and a second kind of particle where the sign of either mass or charge is reversed if $dt/d\tau$ is negative. As these move literally backward in time, we may identify them with Dirac's (1929) antiparticles (Stückelberg, 1941, 1942). So far the standard interpretation has been that we have a negative mass or energy. Indeed this cannot be decided from the equations of motion. But if we look at the electric current generated by the particle,

$$j^\mu(\mathbf{x}) = \frac{\delta S_H}{\delta A_\mu(\mathbf{x})} = q \int d\tau\ \delta^4(\mathbf{x} - \mathbf{x}(\tau))\dot{x}^\mu(\tau)$$

$$\overset{(S)}{=} \sigma q v^\mu(t) \prod_{i=1}^{3} \delta(x^i - x^i(t)) \tag{41}$$

and the corresponding source term for the gravitational field

$$T^{\mu\nu}(\mathbf{x}) = \frac{\delta S_H}{\delta g_{\mu\nu}(\mathbf{x})} = m_0 \int d\tau\ \delta^4(\mathbf{x} - \mathbf{x}(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)$$

$$\overset{(S)}{=} m v^\mu(t) v^\nu(t) \prod_{i=1}^{3} \delta(x^i - x^i(t)) \tag{42}$$

we see that it is the observable charge that depends on the sign σ, not the mass. Hence all particles have positive mass.

There are obviously two transformations that change σ: the inversion of either time or proper time. We first discuss time inversion (Stückelberg, 1941; Feynman, 1949), assuming the transformation exists at least locally also in the context of general relativity. The complete coordinate transformation in five dimensions is given by

$$\begin{aligned} x^{0\prime} &= -x^0, \qquad & x^{i\prime} &= x^i, \quad i = 1, \ldots, 4 \\ p_{\hat\mu}{}' &= \hat T_{\hat\mu}{}^{\hat\nu} p_{\hat\nu}, \qquad & \hat g^{\hat\mu\hat\nu\prime} &= \hat T_{\hat\kappa}{}^{\hat\mu} \hat T_{\hat\lambda}{}^{\hat\nu} \hat g^{\hat\kappa\hat\lambda} \end{aligned} \tag{43}$$

where the 5×5 transformation matrix $\hat T$ is given by

$$\hat T = \mathrm{diag}(-1, 1, 1, 1, 1) \tag{44}$$

The action is invariant under this transformation; hence a transformed trajectory $x^{\mu\prime}(\tau)$ is a solution of the equations of motion if $x^\mu(\tau)$ is. If $x^0(\tau) > 0$, then $x^{0\prime}(\tau) < 0$; the transformed trajectory belongs to an antiparticle with observable charge $-q$. We see that the time inversion changes the charge, although it does not change the variable p_4 that contains information on the charge. But the sign of the variable p_0 that defines the energy is changed and following conventional wisdom we should say that the antiparticle has negative energy. But we cannot decide whether this interpretation is correct, because the exterior field changes under the transformation (43), $\hat g'^{0\hat\mu}(x^{\mu\prime}) = -\hat g^{0\hat\mu}(x^\mu)$ for $\hat\mu = 1, 2, 3, 4$. We will now use the τ inversion transformation to *prove* that antiparticles have positive energy. The inversion of the proper time is a reparametrization of trajectories, not a coordinate transformation. It is defined in any geometrical context by

$$\begin{aligned} \tau' &= -\tau + (\tau_1 + \tau_0) \\ (\mathbf{x}'(\tau'), \mathbf{p}'(\tau')) &= (\mathbf{x}(\tau), -\mathbf{p}(\tau)) \end{aligned} \tag{45}$$

It leaves the action invariant. The coordinates and external fields are not changed by this transformation. Hence in any coordinates we have

$$x^\mu(t)' = x^\mu(t) \tag{46}$$

and

$$v^\mu(t)' = v^\mu(t) \tag{47}$$

Thus the observable positions (46) and velocities (47) of the trajectories $\mathbf{x}(\tau)$ and $\mathbf{x}'(\tau')$ are the same, so if $\mathbf{x}'(\tau')$ is a trajectory of a physical particle, this particle has the same observable mass, charge, energy, and momentum as the particle described by $\mathbf{x}(\tau)$, since the external fields are the same and all observable quantities can be read off the trajectory of the particle. This is the important point of the proper time inversion transformation: it excludes the interpretation that antiparticles have negative energy if we start with the assumption that particles have positive energy p_0c. In general, we conclude that the observable momenta of any particle or antiparticle are given by $\sigma\mathbf{p}$. This also explains the behavior of p_4 under time inversion. We are thus forced to distinguish between the variables and the observables of the classical relativistic theory, a concept that has so far been restricted to quantum mechanics.

The observable energy E of any particle is now given by

$$E = \sigma p_0 c = mcg_{0\nu}v^\nu + \sigma q A_0 c \tag{48}$$

with the relativistic mass m given by (32). Due to large values of the potential A_0 the energy may indeed become negative in the classical theory, but this is not linked to the concept of antiparticles, since the sign of $\dot{x}^0$ does not change during the motion. In general the energy of an antiparticle will be positive, just like that of a particle.

From equation (49) we see that mc^2 is the energy only in the absence of external fields. We look at the nonrelativistic limit of $\sigma p_0 c$ where the spatial components of v are small compared to c. Together with the weak field assumption for the gravitational field

$$g_{00} = 1 + \frac{2\Phi}{c^2}, \qquad g_{ii} = -1, \quad i = 1, 2, 3 \tag{49}$$

with gravitational potential Φ we get

$$\sigma p_0 c \simeq m_0 c^2 + \tfrac{1}{2} m_0 v^2 + m_0 \Phi + \sigma q A_0 c \tag{50}$$

which is clearly the energy of a nonrelativistic particle. If the external fields are time independent, $\sigma p_0 c$ is conserved, as we already pointed out, whereas mc^2 in general is not. We thus have shown that the concept of antiparticles may be included into classical relativistic theory without the feature of negative energy. This means that in the classical theory the process

$$e \to \bar{e} + 2\gamma \tag{51}$$

where an electron changes into a positron under the production of two photons is forbidden by energy conservation if we interpret the arrow as coordinate timelike. Viewed as a process in proper time, the process exists and describes the total reflection of a particle in time, that is, the creation or annihilation of an electron–positron pair.

REFERENCES

Anderson, J. L. (1967). *Principles of Relativity Physics*, Academic Press, New York.
Arzeliès, H. (1972). *Relativistic Point Dynamics*, Pergamon Press, Oxford.
Dirac, P. A. M. (1929). *Proceedings of the Royal Society of London A*, **127**, 360.
Dirac, P. A. M. (1950). *Canadian Journal of Mathematics*, **2**, 129.
Dirac, P. A. M. (1958). *Proceedings of the Royal Society of London A*, **246**, 326.
Doughty, N. A. (1990). *Lagrangian Interaction*, Addison-Wesley, Sydney.
Einstein, A., and Bergmann, P. (1938). *Annals of Mathematics*, **39**, 685.
Falk, G. (1952). *Zeitschrift für Physik*, **132**, 44.
Feynman, R. P. (1949). *Physical Review*, **76**, 749.
Kalman, G. (1962). *Nuovo Cimento*, **23**, 908.
Kaluza, Th. (1921). *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physik, Mathematik*, **54**, 966.
Klein, O. (1926). *Nature*, **118**, 516.
Linder, A. (1965). *Supplemento del Nuovo Cimento*, **3**, 118.
Macke, W. (1952). *Zeitschrift für Naturforschung*, **7a**, 76.
Rund, H. (1962). *Nuovo Cimento*, **23**, 227.
Sauter, F. (1959). *Zeitschrift für Physik*, **156**, 275.
Schay, G. (1962). *Supplemento del Nuovo Cimento (10)*, **26**, 291.
Souriau, J.-M. (1963). *Nuovo Cimento*, **30**, 565.
Stephani, H. (1962). *Wissenschaftliche Zeitschrift*, **11**, 13.
Stückelberg, E. C. G. (1941). *Helvetica Physica Acta*, **14**, 588.
Stückelberg, E. C. G. (1942). *Helvetica Physica Acta*, **15**, 23.
Weinberg, S. (1983). *Physics Letters B*, **125**, 265.